Aras "Homotopy Transfer"

References:

Vallette – "Algebra + Homotopy = Operad"

Chuanic – "On the Perturbation Lemma …" //

Fix <u>homotopy data</u>:

$$\begin{tikzcd} (A, d_A) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H, d_H) \arrow[l, shift left, "i"] \end{tikzcd}$$

cochain complexes & i, p cochain maps

& h a homotopy $i \circ p \sim id_A$

(Vague)
<u>Question</u>: What structures can be transferred along this data?

"structure" might be, e.g., associative product on A

<u>Example</u> "perturbing the differential"

Let $\delta : A \to A$ be a map of same degree as d_A s.t.

- $(d_A + \delta)^2 = 0$

- $1 - \delta h$ is invertible

Then we obtain "new" homotopy data:

$$\begin{tikzcd} (A, d_A + \delta) \arrow[loop left, "h' = h + h(1-\delta h)^{-1}\delta h"] \arrow[r, shift left, "p + p(1-\delta h)^{-1}\delta h \, = p'"] & (H, d_H + p(1-\delta h)^{-1}\delta i) \arrow[l, shift left, "\underset{i'}{\shortparallel}\; i + h(1-\delta h)^{-1}\delta i"] \end{tikzcd}$$

Moreover, if i is a quasi-iso, then so is i'.

__Pf__ Just calculate (see Crainic). $\square$

What if A is a (non-unital) dga?

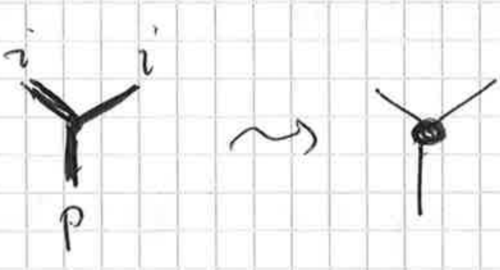
$\nu : A \otimes A \longrightarrow A$ is multiplication

__picture__: 

input $\downarrow$ output $\qquad$ $A \otimes A \downarrow A$

We can define a "candidate multiplication" on H:

$$\mu_2 : H \otimes H \xrightarrow{i \otimes i} A \otimes A \xrightarrow{\nu} A \xrightarrow{p} H$$

__picture__: 

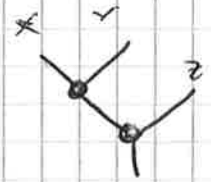
$H \otimes H \downarrow H$

__Moral__: You shouldn't expect this μ_2 to be strictly associative b/c H is not strictly equivalent to A but only homotopic

Let's try: $x, y, z \in H$

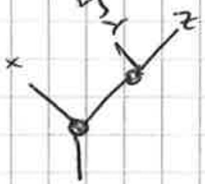
there's no reason for equality btw

$p(i(p(i(x)\cdot i(y)))\cdot i(z))$



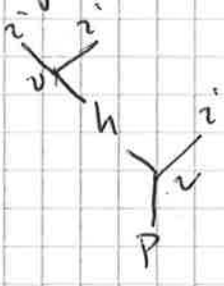
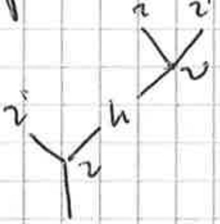
&

$p(i(x)\cdot i(p(i(y)\cdot i(z))))$



but

is null-homotopic as a map $H^{\otimes 3} \to H$

Set $\mu_3 :=$  $-$  $\cong:$ 

so $\mu_3 : H^{\otimes 3} \to H$ of deg $+1$

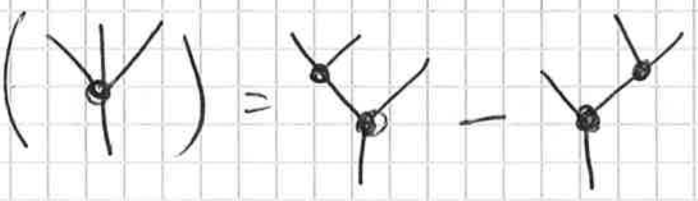
(same degree as h)

[We are homological]

If $i \circ p = id_A$ & $h \equiv 0$, then μ_3 would be zero.

But even if μ_2 is associative, μ_3 need not be zero

In fancier terms:

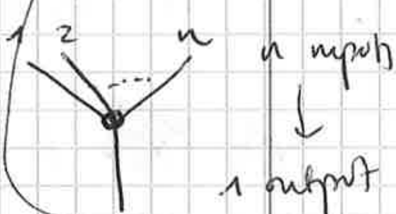
$d_{\mathrm{Hom}(H^{\otimes 3}, H)}$ 

... higher associativity relations are encoded in higher "multiplication maps"

<u>Def</u> An <u>A_∞ algebra</u> is a chain complex (X, d_X) equipped with maps

$$\gamma_n : X^{\otimes n} \longrightarrow X$$

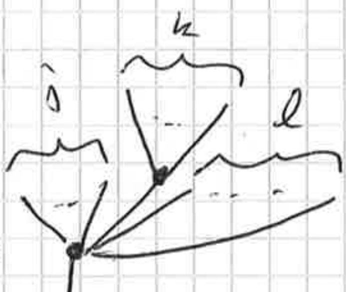
<u>picture</u>



of degree $n-2$, $n \geq 2$, such that

$$d_{\mathrm{Hom}(X^{\otimes n}, X)}(\gamma_n) = \cancel{\sum_{\substack{1<k<n \\ j+k+l=n}} (-1)^{jk+l}}$$

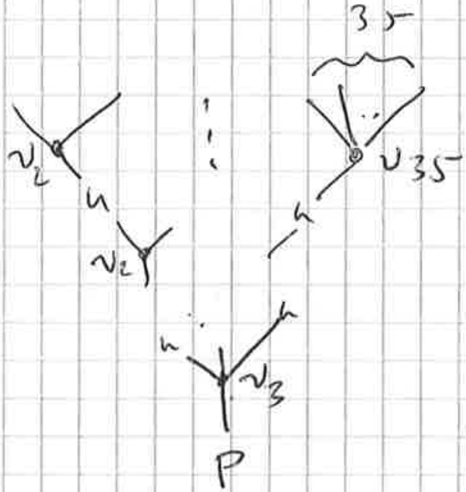
$$= -\sum_{\substack{1<k<n \\ j+k+l=n}} (-1)^{jk+l} \gamma_{j+l+1} \circ \Big(\underbrace{id \otimes \cdots \otimes id}_{j \text{ times}} \otimes \gamma_k \otimes \underbrace{id \otimes \cdots \otimes id}_{l \text{ times}} \Big)$$

$= -\sum_{\dots} \pm$ 

Prop Assume A is equipped with an A_∞ algebra structure $(\nu_n : A^{\otimes n} \to A)_{n \geq 2}$

Then

$$\left(\mu_n = \sum_{\substack{\text{planar rooted} \\ \text{trees w/} \\ n \text{ leaves}}} \pm \right.$$



$$\Bigg)_{n \geq 2}$$

defines an A_∞ algebra structure on H.

Remark: dg algebras correspond to A_∞-algebra structures such that $\nu_n = 0$, $n \geq 3$.

Cor If A is a dga, then H has a "transferred" A_∞-algebra structure.

Example Let A be a dga such that there are splittings

$$A_n \cong \ker(d_n) \oplus \operatorname{im}(d_n) = Z_n \oplus B_{n-1}$$

$$Z_n \cong H_n(A) \oplus \ker(d_{n-1}) = H_n(A) \oplus B_n$$

Then we get homotopy data : $h \circlearrowright (A, d) \underset{i}{\overset{p}{\rightleftarrows}} (H_* A, 0)$

inclusion $i : H_n(A) \hookrightarrow A_n$

projection $p_n : A_n \longrightarrow H_n(A)$

homotopy $h_n : A_n \twoheadrightarrow B_n \hookrightarrow A_{n+1}$

Then you can transfer dga structure on A to an A_∞ alg str. on $H_* A$

Special case : X topological space, k a field

$$A = C^*_{sing}(X; k) \text{ w/ cup product}$$

The transferred structure on $H^*_{sing}(X; k)$ has

μ_2 = cup product on cohomology

μ_n, $n \geq 2$ means what? (explain later)

Special case M smooth compact oriented manifold w/ Riemannian metric g

Theorem (Hodge)

There is an isomorphism

$$\Omega^*(M) \supseteq \underset{\text{"harmonic forms"}}{\mathcal{H}^*_g(M)} \xrightarrow{\cong} H^*_{dR}(M)$$

and homotopy data

$$G \circlearrowright (\Omega^*(M), d) \underset{i}{\overset{p}{\rightleftarrows}} (H^*_{dR}(M), 0)$$

Now $\Omega^*(M)$ has the <u>wedge product</u>

Then
- μ_2 = wedge product on cohomology
- μ_n, $n \geq 3$ means what?

These higher μ_n's explain Massey products

<u>Def</u> For a dga A with the required splittings, we call the transferred multiplications (μ_n) <u>A_∞ Massey products</u>.

<u>Def</u> The <u>"classical" (triple) Massey product</u> is:

A a dga, $[u], [v], [w] \in H_* A$ s.t.

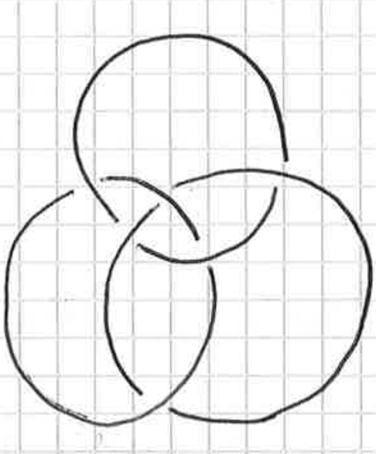
$$[u][v] = 0 = [v][w]$$

their Massey product is

$$\langle [u], [v], [w] \rangle = \left\{ \left[(-1)^u u \cdot x + (-1)^y y \cdot w \right] : dx = (-1)^v vw,\ dy = (-1)^u u \cdot v \right\}$$

$$\in H_*(A) / ([u] \cdot H_* A + H_* A [w])$$

Standard example:

Borromean rings $R =$  $\subset S^3$

Any pair of knots is unlinked

but the cocycles representing each knot in

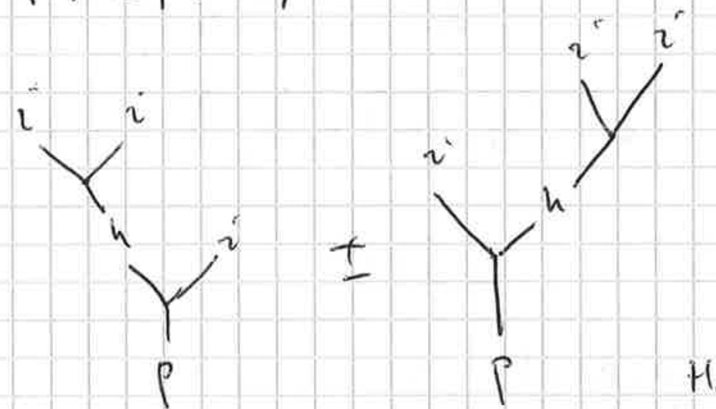
$S^3 \setminus R$ have nontrivial triple Massey product

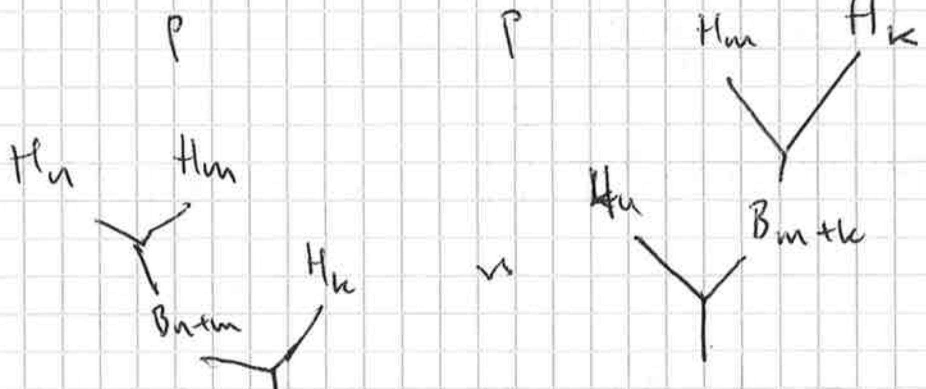
<u>Prop</u> In the previous set-up,

$\mu_3([u], [v], [w])$ is a representative

for $\langle [u], [v], [w] \rangle$

<u>Picture</u>





Note: The classical Massey product lives in a quotient of $H_* A$, but A_∞-alg Massey product lives in $H_* A$. The A_∞ alg str depends on a choice of splitting of A, but those choices all are equivalent & induce same answer on the quotient.

$\Downarrow$

classical answer!

Prop If A is equipped with the structure of an L_∞ algebra structure $(l_n : A^{\otimes n} \to A)_{n \geq 2}$ this yields an L_∞ structure on H.

In particular, for V a dgla structure over a field k, a choice of splitting produces an L_∞ alg str on $(H_* V, 0)$.